Proof of the GM–GR parity theorem for the two-body problem

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Abstract: One loosely defines Mechanics as a physical theory that rests on the concepts of mass and force and a law of inertia. In contrast, one loosely defines General Relativity as a physical theory that describes how mass and energy curve spacetime, causing objects to move along the straightest possible paths within that curved geometry. For a long time, scientists viewed Mechanics and General Relativity as fundamentally irreconcilable theories, with neither being a mere modification of the other, but rather grounded in distinct and incompatible physical principles. This theorem reshapes that understanding by proving that a modified Mechanics, called General Mechanics, fully aligns with General Relativity in the two-body problem. The trajectories in both theories are the same, and it follows that both adopt the same physical principles. NOTE FROM THE EDITOR-IN-CHIEF: In a blinded assessment, I asked five scientists to verify the mathematics of the parity theorem presented in this article before the article would undergo a review. All of them verified the mathematics. I took this additional step because of the theorem's potentially significant impact on the fields of Mechanics and Relativity. © *2025 Physics Essays Publication*. [http://dx.doi.org/10.4006/0836-1398-38.2.122]

Résumé: La mécanique est grossièrement décrite comme une théorie physique reposant sur les concepts de masse et de force ainsi que sur le principe d'inertie. Quant à elle, la relativité générale est dépeinte comme une théorie physique décrivant comment la masse et l'énergie courbent l'espace-temps, amenant les objets à se déplacer selon les trajectoires les plus droites possibles dans cette géométrie courbée. Pendant longtemps, les scientifiques ont considéré la mécanique et la relativité générale comme des théories fondamentalement irréconciliables, aucune n'étant une simple modification de l'autre, mais plutôt fondées sur des principes physiques distincts et incompatibles. Le théorème proposé ici remanie cette compréhension en démontrant qu'une mécanique modifiée, appelée mécanique générale, s'aligne pleinement avec la relativité générale dans le cas du problème à deux corps. Les trajectoires prédites par les deux théories sont identiques, ce qui implique qu'elles reposent sur les mêmes principes physiques. NOTE DU RÉDACTEUR EN CHEF: Lors d'une évaluation en double aveugle, j'ai demandé à cinq scientifiques de vérifier la démonstration mathématique du théorème de parité présenté dans cet article avant qu'il ne soit soumis à une évaluation. Tous ont validé cette démonstration. J'ai pris cette mesure de validation supplémentaire en raison de l'impact potentiellement significatif de ce théorème sur les domaines de la mécanique et de la relativité.

Key words: Electromagnetism; Field; Fragment of Energy; General Mechanics; General Relativity; Interaction Force; Mechanics; Relativity; Schwarzschild; Spacetime.

I. INTRODUCTION

Mechanics, loosely defined, is a physical theory based on mass, force, and inertia, whereas General Relativity (GR) describes how mass and energy curve spacetime, guiding motion along the straightest paths in that curvature. Over the last one hundred years, their outwardly antithetical appearances led scientists to believe they are irreconcilable, though they never could prove this.[b] Not only was this irreconcilability never proven, this article proves for the two-body problem that it is false—a revelation that will surprise scientists and the populous at large. Moreover, the article presents the precise modification to Mechanics that makes it in full agreement with General Relativity for the two-body problem. Throughout the article, we have referred to the modified Mechanics as General Mechanics (GM).

In the next two sections, we state the Theorem and give its proof.[c] The Theorem starts with two sets of three govern-

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[b]The two major arguments made for the two theories being irreconcilable were (1) the principle of equivalence claim, based on the paradox that inertial mass and gravitational mass are inexplicably equal, and (2) the inverse square law claim, which points to an instantaneous gravitational effect over a finite distance. Other claims were (3) the limitation claim, arguing that the concept of force is for weak fields only, (4) the relativity claim, which points to the use of the inertial frame in mechanics as violating relativity, and (5) the least action claim, which applies the principle of least action in general relativity instead of the principle of inertia. For example, one finds claim one in Ref. 1 (p. 148), Ref. 2 (pp. 42, 111, and 113), Ref. 3 (pp. 117 and 120), Ref. 4 (pp. 8–11 and 65), and Ref. 5 (pp. 11, 15–17, 20, and 21). Claim two is in Ref. 1 (p. 147), Ref. 2 (pp. 107 and 131), Ref. 4 (pp. 58–60), and Ref. 6 (pp. 4 and 177). Claim three is in Ref. 1 (p. 149) and Ref. 4 (p. 72). Claim four is in Ref. 2 (p. 107) and Ref. 5 (pp. 10 and 114). Finally, one finds claim five made in Ref. 2 (p. 170).

[c]We first published this theorem in an appendix of an engineering journal,[7] where the theorem itself received little scrutiny, and here we have published the proof in a standalone physics journal purposefully reviewed by a large number of physicists and readily available for the readers to judge for themselves.

ing equations, the first set for GM and the second set for GR. We formulated the two-body problem in each set as a one-body problem expressed in terms of polar coordinates because this is how Schwarzschild formulated it for GR.[8] For completeness, Appendix A gives the development of the conversion of a two-body problem into a one-body problem. The Theorem proves, with identical initial conditions, that the two theories produce identical trajectories.

The three governing equations for GM consist of Eq. (1), which is the Minkowski spacetime metric in polar coordinates, Eq. (2), which is Newton's general law of inertia[d)] in polar coordinates, and Eq. (3), which is the general universal law of gravitation in polar coordinates. Equation (2) also specifies the relativistic expressions for velocity and acceleration, derived from Eq. (1). For completeness, Appendix B gives those derivations. The third equation is the universal law of gravitation now generalized so it accommodates relativistic motion.[e)] Equations (1)–(3) along with the initial conditions are sufficient to predict fully the trajectory of a body.

The three governing equations for GR consist of Eq. (4), which is the Schwarzschild metric in polar coordinates, Eq. (5), which is the equation governing the spatial behavior of a body in polar coordinates, and which one obtains from the GR apparatus, and Eq. (6), which is the expression for angular momentum.[1] The GR community well knows all three equations and, with the initial conditions, they predict fully the trajectory of a body.

The GM–GR parity theorem states that the trajectories determined from the set of GM equations are identical to the trajectories determined from the set of GR equations. Again, Sec. II states the Theorem, and Sec. III gives the proof. Section IV closes with several remarks.

[d)]Equations (1) and (2) originate from the Theory of Special Relativity.

[e)]The discovery of the general universal law of gravitation was largely a fortunate stroke of serendipity. In early 2020, with extra time on our hands due to the Covid-19 epidemic, we decided to attempt to reproduce results in GR by modifying the universal law of gravitation to account for relativistic effects, unsure whether others before us would have already tried our approach (See Einstein's attempt at this[9]). The first author's focus was theoretical, and the second author's focus was numerical. We started by examining the precession of Mercury as it orbits the Sun because this problem yields exact solutions in GR to which we could compare our solutions. Karl Schwarzschild (1873–1916) discovered them in 1916, in the last year of his life, one year after Albert Einstein (1879–1955) introduced GR. Anyway, based on various considerations, we hypothesized that a modification to the law that might reproduce GR would be a multiplication factor of the analytical form $B = 1 + f(r,h)$. The goal was to find the function $f(r,h)$, in which r is the distance between Mercury and the Sun and h is the specific relativistic angular momentum of the Mercury–Sun system about its mass center. We empirically determined that $f(r,h) = 3(h/cr)^2$. The precession of Mercury that we found agreed numerically within three decimal places to the precession in GR from the Schwarzschild solution. We then examined the problem of light traveling past the Sun using the same relativistic universal law of gravitation. With it, we found numerical agreement with the Schwarzschild solution, again, to three decimal places, and published these results in 2020.[10] The results in that article were for weak gravitational fields, and soon after, we verified agreement in strong gravitational fields, too, concluding that we had stumbled on a mechanics formulation of the theory of general relativity.[11] We first referred to it as the theory of spacetime impetus, given that we were applying the new relativistic universal law of gravitation in spacetime, with its spacetime or relativistic concept of impetus (inertia). We were convinced that this new formulation is in full agreement with GR but we were lacking a mathematical proof. After that, we focused on seeking the mathematical proof presented in this article.

II. STATEMENT OF THE GM–GR PARITY THEOREM

Let $()' \triangleq d()/d\tau$, let Eqs. (1)–(3) govern GM, and[f)] Eqs. (4)–(6) govern GR:

General Mechanics (GM)

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2, \tag{1}$$

$$F_{Rr} = \mu a_{Rr}, \quad F_{R\phi} = \mu a_{R\phi}, \quad \mu \triangleq \frac{m_a m_b}{m_a + m_b}$$
$$v_{Rr} \triangleq r', \quad v_{R\phi} \triangleq r\phi', \quad a_{Rr} \triangleq r'' - r\phi'^2, \quad a_{R\phi} \triangleq r\phi'' + 2r'\phi', \tag{2}$$

$$F_{Rr} = -\frac{G m_a m_b}{r^2}\left(1 + 3\left(v_{R\phi}/c\right)^2\right), \quad F_{R\phi} = 0. \tag{3}$$

General Relativity (GR)

$$c^2 d\tau^2 = c^2(1 - r_s/r)dt_G^2 - (1 - r_s/r)^{-1} dr^2 - r^2 d\phi^2, \quad r_s \triangleq \frac{2GM}{c^2}, \tag{4}$$

$$\frac{d^2u}{d\phi^2} + u - \frac{3GM}{c^2}u^2 = \frac{GM}{h^2}, \quad u \triangleq \frac{1}{r}, \quad M \triangleq m_a + m_b, \tag{5}$$

$$h = r v_{R\phi} = \text{constant}. \tag{6}$$

The trajectories $r(\tau)$ and $\phi(\tau)$ determined from the set of equations (1)–(3) and independently from the set of equations (4)–(6) are identical when prescribed with identical initial conditions $r(0), r'(0), \phi(0)$, and $\phi'(0)$.

III. PROOF

We prove the parity of GM and GR by deriving Eqs. (4)–(6) from Eqs. (1)–(3). First, recognize that the spatial coordinates r and ϕ and the proper time τ are the same in both sets of equations and that the coordinate time t in GM and the coordinate time t_G in GR are different. Given that the coordinate time t in GM and the coordinate time t_G in GR are different, the first step taken is to determine whether there exists an analytical relationship between them. One

[f)]The term parity theorem refers to a theorem that shows that two sets of variables and relationships (theories) are on par with each other, that is, that there exists a one-to-one correspondence between the variables and the relationships in each set. Pictorially, the one-to-one correspondence is

$$\begin{array}{ccc} GM & & GR \\ \begin{pmatrix} \tau & r \\ \phi & t \end{pmatrix} & \leftrightarrow & \begin{pmatrix} \tau & r \\ \phi & t_G \end{pmatrix}. \end{array}$$

In both GM and GR, the geometric variables that one measures directly are the proper time τ, the radial coordinate r, and the circumferential angle ϕ; they are the same in both theories, as shown. In addition, GM defines a coordinate time t and GR defines a coordinate time t_G that differ. Not depicted, the relationships in GM are Eqs. (1)–(3), and in GR they are Eqs. (4)–(6). Also not depicted, both theories share the independent parameters $G, m_a, m_b, and\ c$. The proof first determines the one-to-one correspondences between t and t_G ($t_G(t)$ and $t(t_G)$). Toward this end, the proof first finds the functions dt_G/dt and dt/dt_G. They establish the one-to-one correspondences sought because they lead to $t_G(t) = t_G(0) + \int_0^t (dt_G/dt)dt$ and $t(t_G) = t(0) + \int_0^t (dt/dt_G)dt_G$. After that, the proof determines the correspondences between the relationships in GM and GR.

confirms this by deriving the gravitational coordinate time factor dt_G/dt as follows:

1. Equate the proper time increments $d\tau$ in Eqs. (1) and (4),

$$c^2dt^2 - dr^2 - r^2d\phi^2 = \left(1-\frac{r_s}{r}\right)c^2dt_G^2 - \left(1-\frac{r_s}{r}\right)^{-1}dr^2 - r^2d\phi^2.$$

2. Cancel the term $-r^2d\phi^2$,

$$c^2dt^2 - dr^2 = \left(1-\frac{r_s}{r}\right)c^2dt_G^2 - \left(1-\frac{r_s}{r}\right)^{-1}dr^2.$$

3. Divide by c^2dt^2 and solve for $(dt_G/dt)^2$,

$$1-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2 = \left(1-\frac{r_s}{r}\right)\left(\frac{dt_G}{dt}\right)^2 - \frac{1}{c^2}\left(1-\frac{r_s}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2,$$

$$\begin{aligned}\left(\frac{dt_G}{dt}\right)^2 &= \frac{1}{1-\frac{r_s}{r}}\left[1-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2+\frac{1}{c^2}\left(1-\frac{r_s}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2\right],\\ &= \frac{1}{1-\frac{r_s}{r}}\left[1-\frac{1}{c^2}\left(-1+\frac{1}{1-\frac{r_s}{r}}\right)\left(\frac{dr}{dt}\right)^2\right].\end{aligned}$$

4. Take the square root and form a common denominator,

$$\frac{dt_G}{dt} = \left\{\frac{1}{1-\frac{r_s}{r}}\left[1+\frac{1}{c^2}\left(\frac{\frac{r_s}{r}}{1-\frac{r_s}{r}}\right)\left(\frac{dr}{dt}\right)^2\right]\right\}^{1/2}. \tag{7}$$

The gravitational time factor, Eq. (7), transforms coordinate time t in GM to coordinate time t_G in GR where $r > r_s$. The inverse dt/dt_G of the gravitational time factor exists when $r > r_s$, too.[g)]
Equation (7) shows how t in GM and t_G in GR are related. Equations (7) and (1) yield Eq. (4). Alternatively, Eqs. (7) and (4) yield Eq. (1). It remains to derive Eqs. (8) and (9) from Eqs. (1)–(3).

5. Substitute Eq. (3) into Eq. (2), cancel terms, and invoke $M \triangleq m_a + m_b$,

$$a_{Rr} = -G\frac{M}{r^2}\left(1+3\left(\frac{v_{R\phi}}{c}\right)^2\right), \quad a_{R\phi} = 0.$$

6. From step 5 and Eq. (2),

$$r'' - r\phi'^2 = -G\frac{M}{r^2}\left(1+3\left(\frac{v_{R\phi}}{c}\right)^2\right),$$
$$0 = r\phi'' + 2r'\phi'.$$

7. The specific angular momentum is the same in GM and GR,
$h = rv_{R\phi}$.
Differentiate h with respect to proper time, invoke the product and chain rules, and from the second equation in step 6 and from Eq. (2),

$$h = r^2\phi', \tag{8}$$
$$h' = \left(r^2\phi'\right)' = r^2\phi'' + 2rr'\phi' = r\left(r\phi''+2r'\phi'\right) = 0.$$

Thus, h is constant, as Eq. (6) requires. Equation (8) and step 7 have yielded Eq. (6). It remains to prove Eq. (5) from Eqs. (1)–(3).

8. Differentiate u with respect to ϕ, invoke the chain rule, Eqs. (5) and (8),

$$\frac{du}{d\phi} = \frac{d}{d\phi}r^{-1} = -r^{-2}\frac{dr}{d\phi} = -r^{-2}\frac{dr}{d\tau}\frac{d\tau}{d\phi} = -\frac{1}{h}\frac{dr}{d\tau}.$$

9. Again, differentiate with respect to ϕ,

$$\begin{aligned}\frac{d^2u}{d\phi^2} &= \frac{d}{d\phi}\left(-\frac{1}{h}\frac{dr}{d\tau}\right) = -\frac{1}{h}\frac{d\tau}{d\phi}\frac{d}{d\tau}\left(\frac{dr}{d\tau}\right) = -\frac{1}{h}\frac{d\tau}{d\phi}\frac{d^2r}{d\tau^2}\\ &= -\frac{1}{r^2}\left(\frac{d\tau}{d\phi}\right)^2\frac{d^2r}{d\tau^2}.\end{aligned}$$

10. From step 9,

$$\begin{aligned}\frac{d^2u}{d\phi^2}+u &= -\frac{1}{r^2}\left(\frac{d\tau}{d\phi}\right)^2\frac{d^2r}{d\tau^2}+\frac{1}{r}\\ &= -\frac{1}{r^2}\left(\frac{d\tau}{d\phi}\right)^2\left(\frac{d^2r}{d\tau^2}-r\left(\frac{d\phi}{d\tau}\right)^2\right)\\ &= -\frac{1}{r^2}\left(\frac{d\tau}{d\phi}\right)^2a_{Rr}.\end{aligned}$$

11. From steps 10 and 5,

$$\begin{aligned}\frac{d^2u}{d\phi^2}+u &= \frac{1}{r^2}\left(\frac{d\tau}{d\phi}\right)^2G\frac{M}{r^2}\left(1+3\left(\frac{v_{R\phi}}{c}\right)^2\right)\\ &= \left(\frac{1}{r^2}\frac{d\tau}{d\phi}\right)^2GM\left(1+3\left(\frac{v_{R\phi}}{c}\right)^2\right),\\ &= \frac{GM}{h^2}\left(1+3\left(\frac{h}{rc}\right)^2\right) = \frac{GM}{h^2}+\frac{3GM}{c^2}u^2,\end{aligned}$$

$$\frac{d^2u}{d\phi^2}+u-\frac{3GM}{c^2}u^2 = \frac{GM}{h^2}. \tag{9}$$

Equation (9) is identically Eq. (5). Thus, Eqs. (4)–(6) are on par with Eqs. (1)–(3), where $r > r_s$.

IV. REMARKS

The parity of GM and GR leads one to reassess mechanics and relativity. Below, we provide several remarks about

[g)]Although the parity [Eq. (7)] only applies to $r > r_s$, solutions to the problem when $r < r_s$ exist, too. Furthermore, the GM solution has no singularity when crossing $r = r_s$, and the GR solution has a singularity when crossing $r = r_s$.

physical theories and the impact of this theorem on unification.

A. Geometric theories

One says that the theory of general relativity (GR) is a geometric theory.[h)] This characterization captures a number of the theory's stunning features. For one, GR is a geometric theory because it describes the interaction between bodies by a geometry's curvature instead of by the conception of force.[12] It retains the kinematic aspects of modern physical theories while abandoning its kinetic aspects. Second, one might regard GR to be a geometric theory, not just because of its curved spacetime, but also because of its abandonment of the need for laws that govern forces, replacing them with the very elegant idea of equating the trajectory of a body and the curved geometry's straight line (geodesic). Finally, one could regard GR to be a geometric theory because, by convention, physical measurements are fundamentally geometric. In the two-body problem treated in this article, the radial coordinate r, the circumferential angle ϕ, and the proper time τ constitute physical measurements. Such concepts as the gravitational force and energy are mathematical devices obtained together with the introduction of such physical constants as mass m, universal gravitational constant G, and speed of light limit c. However, this reason applies to every physical theory, which by itself, would make every physical theory a geometric theory.

B. Curved geometries as a physical principle

This brings us back to the question of the distinction between GR with its curved Riemannian geometries and GM with its flat Euclidean geometry, about whether the curved geometries in GR introduce any new physical principles on top of constituting a different mathematical formulation of a physical theory.

Historically, Nonrelativistic or Newtonian Mechanics (NM)[13] and electromagnetism (EM)[14] employed the classical principles of inertia and light. The modern principles began with the Theory of Special Relativity (SR)[15] by introducing a modern principle of light expressed by an ordinary (Minkowski) spacetime metric. The principles of inertia and light accurately predict the trajectories of light and massive bodies, except in the case of bodies traveling at near-light speeds under the influence of gravitation. In those cases, SR erroneously predicts that gravitation has no influence on the trajectories of bodies. GR remedied that shortcoming but also raised the question about the role of curved geometries. Were curved geometries a new physical principle in themselves or were they a mathematical formulation in the first theory to unite the modern principles of inertia, gravitation, and light? The GM–GR parity theorem shows us that the curved geometry for the two-body problem does not introduce a new physical principle. We found its straight trajectories (geodesics) in its curved geometry to be equivalent to its curved trajectories in a flat (straight) geometry. We found that the shortcoming rests in SR not having updated the universal law of gravitation to accommodate relativistic effects.

C. Unification

The GM–GR parity theorem advances unification. Over the last hundred years, scientists struggled to unite GR, the one theory that addresses the principles of inertia, gravitation, and light, with the theories that predict observations of behavior at the other scales, thinking incorrectly that curved geometries play a necessary role in unification and that mechanics would be inadequate without them. We now know that a more general mechanics, one that invokes the modern principle of light but does not invoke curved geometries, can accommodate this goal, too, and can return analysts to a problem-solving approach that is both intuitive and familiar.

NOMENCLATURE

$\boldsymbol{a}_R \triangleq \boldsymbol{a}_{Rb} - \boldsymbol{a}_{Ra} =$ Relativistic acceleration vector in the one-body problem

$\boldsymbol{a}_{Ra}, \boldsymbol{a}_{Rb} =$ Relativistic acceleration vectors of bodies a and b

$a_{Rr}, a_{R\phi} =$ Radial and circumferential components of relativistic acceleration vector

$c =$ Speed of light

$F_R \triangleq F_{Rab} =$ Interaction force vector in the one-body problem

$F_{Rab}, F_{Rba} =$ Interaction force vector by body a on body b, and vice versa

$F_{Rr}, F_{R\phi} =$ Radial and circumferential components of the force vector

$G, r_s \triangleq 2GM/c^2 =$ Universal gravitational constant, Schwarzschild radius[i)]

$h =$ Specific relativistic angular momentum

$m_a, m_b =$ Masses of bodies a and b

$r, \phi =$ Radial coordinate, circumferential angle

$\boldsymbol{v}_R \triangleq \boldsymbol{v}_{Rb} - \boldsymbol{v}_{Ra} =$ Relativistic velocity vector in the one-body problem

$\boldsymbol{v}_{Ra}, \boldsymbol{v}_{Rb} =$ Relativistic velocity vectors of bodies a and b

$\boldsymbol{v}_{RC} =$ Relativistic velocity vector of the mass center

$v_{Rr}, v_{R\phi} =$ Radial and circumferential components of the relativistic velocity vector

$\mu, M =$ Reduced mass, combined mass

$\tau, t, t_G =$ Proper time, coordinate time in GM, coordinate time in GR

APPENDIX A: THE TWO-BODY PROBLEM

Figure 1 shows the two-body problem and the one-body problem. In the two-body problem, the bodies are a distance r from each other, and they are each moving. In the one-body problem, the body is on the right a distance r from the origin.

[h)] We are using the term geometry in the broad context that includes the curved geometry whose dimensions are spatial and temporal in addition to the ordinary context of the Euclidean geometry whose dimensions are spatial.

[i)] In the definition of the Schwarzschild radius, one often assumes that body a is stationary, neglecting the mass of body b for which $M = m_a$. The definition of the Schwarzschild radius given in Eq. (5) accommodates more broadly the two-body problem (see Appendix A).

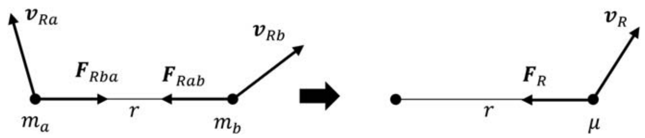

FIG. 1. The two-body problem and the equivalent one-body problem.

Begin with the two-body problem,

$$\boldsymbol{F}_{Rba} = m_a \boldsymbol{a}_{Ra}, \tag{A1a}$$

$$\boldsymbol{F}_{Rab} = m_b \boldsymbol{a}_{Rb}, \tag{A1b}$$

$$\boldsymbol{F}_{Rab} = -\boldsymbol{F}_{Rab}. \tag{A1c}$$

Equations (A1a) and (A1b) govern the motion of body a and body b. From Eq. (A1),

$$\boldsymbol{a}_R = \frac{\boldsymbol{F}_{Rab}}{m_b} - \frac{\boldsymbol{F}_{Rba}}{m_a} = \left(\frac{1}{m_b} + \frac{1}{m_a}\right)\boldsymbol{F}_R = \frac{1}{\mu}\boldsymbol{F}_R,$$
$$\boldsymbol{F}_R = \mu \boldsymbol{a}_R. \tag{A2}$$

Equation (A2) governs the motion of a body in the one-body problem. In addition to the quantities defined in Eqs. (A1) and (A2), one can define how the relativistic velocity vectors in both problems relate to each other,

$$\boldsymbol{v}_R \triangleq \boldsymbol{v}_{Rb} - \boldsymbol{v}_{Ra}, \tag{A3a}$$

$$\boldsymbol{v}_{RC} \triangleq \frac{1}{m}(m_a \boldsymbol{v}_{Ra} + m_b \boldsymbol{v}_{Rb}), \tag{A3b}$$

$$\boldsymbol{v}_{Ra} = -\frac{m_b}{m}\boldsymbol{v}_R + \boldsymbol{v}_{RC}, \tag{A3c}$$

$$\boldsymbol{v}_{Rb} = \frac{m_a}{m}\boldsymbol{v}_R + \boldsymbol{v}_{RC}. \tag{A3d}$$

Equations (A3a) and (A3b) express the relativistic velocity vector of the body in the one-body problem and the relativistic velocity vector of the mass center in the two-body problem in terms of the relativistic velocity vectors of the two bodies, and Eqs. (A3c) and (A3d) express the inverse relationships. Finally, note that the one-body problem and its solution are independent of the velocity of the mass center $\boldsymbol{v}_{RC}$, so the one-body problem applies to the class of problems that have the same $\boldsymbol{v}_R$ and any $\boldsymbol{v}_{RC}$.

APPENDIX B: RELATIVISTIC KINEMATICS

Begin with Eq. (1) given again by

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2. \tag{B1}$$

In Eq. (B1), define the distance increment between two points as $dl \triangleq \sqrt{dr^2 + r^2 d\phi^2}$. Speed is $v \triangleq dl/dt$.

1. Divide Eq. (B1) by $c^2 dt^2$,

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \left(\frac{v}{c}\right)^2.$$

2. Define the Lorentz factor,

$$\gamma \triangleq \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}. \tag{B2}$$

In polar coordinates, the position vector, the velocity vector, and the acceleration vector are

$$\boldsymbol{r} \triangleq r\boldsymbol{n}_r, \tag{B3a}$$

$$\boldsymbol{v} \triangleq \frac{d\boldsymbol{r}}{dt} = v_r \boldsymbol{n}_r + v_\phi \boldsymbol{n}_\phi, \tag{B3b}$$

$$\boldsymbol{a} \triangleq \frac{d\boldsymbol{v}}{dt} = a_r \boldsymbol{n}_r + a_\phi \boldsymbol{n}_\phi. \tag{B3c}$$

Above, $\boldsymbol{n}_r$ is the radial unit vector, and $\boldsymbol{n}_\phi$ is the circumferential unit vector. The relativistic velocity vector and the relativistic acceleration vector are

$$\boldsymbol{v}_R \triangleq \frac{d\boldsymbol{r}}{d\tau} = v_{Rr}\boldsymbol{n}_r + v_{R\phi}\boldsymbol{n}_\phi, \tag{B4a}$$

$$\boldsymbol{a}_R \triangleq \frac{d\boldsymbol{v}_R}{d\tau} = a_{Rr}\boldsymbol{n}_r + a_{R\phi}\boldsymbol{n}_\phi. \tag{B4b}$$

3. From Eqs. (B3b), (B4a), and (B2), the relationship between the velocity vector and the relativistic velocity vector is

$$\boldsymbol{v}_R = \frac{dt}{d\tau}\frac{d\boldsymbol{r}}{dt} = \gamma \boldsymbol{v}. \tag{B5}$$

Next, develop the relationship between the acceleration vector and the relativistic acceleration vector:

4. Differentiate $\boldsymbol{v}_R$ in Eq. (B5) with respect to proper time and invoke the chain and product rules and Eq. (B5),

$$\boldsymbol{a}_R = \frac{d\boldsymbol{v}_R}{d\tau} = \frac{dt}{d\tau}\frac{d(\gamma\boldsymbol{v})}{dt} = \gamma\left(\frac{d\gamma}{dt}\boldsymbol{v} + \gamma\boldsymbol{a}\right). \tag{B6}$$

5. Determine an expression for the coordinate time derivative $d\gamma/dt$ of the Lorentz factor considering Eq. (B2) and that $v^2 = \boldsymbol{v}\cdot\boldsymbol{v}$,

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1 - \frac{\boldsymbol{v}\cdot\boldsymbol{v}}{c^2}\right)^{-1/2}$$
$$= -\frac{1}{2}\left(1 - \frac{\boldsymbol{v}\cdot\boldsymbol{v}}{c^2}\right)^{-\frac{3}{2}}\left(-\frac{2\boldsymbol{v}\cdot\boldsymbol{a}}{c^2}\right) = \gamma^3\frac{\boldsymbol{v}\cdot\boldsymbol{a}}{c^2}. \tag{B7}$$

6. Finally, substitute Eq. (B7) into Eq. (B6) and rearrange terms,

$$\boldsymbol{a}_R = \gamma\left(\frac{d\gamma}{dt}\boldsymbol{v} + \gamma\boldsymbol{a}\right) = \gamma\left(\gamma^3\boldsymbol{v}\frac{\boldsymbol{v}\cdot\boldsymbol{a}}{c^2} + \gamma\boldsymbol{a}\right)$$
$$= \gamma^2\left(\boldsymbol{a} + \frac{\gamma^2}{c^2}(\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{v}\right). \tag{B8}$$

Equation (B8) expresses the relationship between the relativistic acceleration vector and the acceleration vector. Next, derive the inverse relationship.

7. Take the dot product of $\boldsymbol{v}$ and $\boldsymbol{a}_R$ in Eq. (B8), form a common denominator, and consider Eq. (B2),

$$\boldsymbol{v}\cdot\boldsymbol{a}_R=\boldsymbol{v}\cdot\gamma^2\left(\boldsymbol{a}+\frac{\gamma^2}{c^2}(\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{v}\right)=\gamma^2(\boldsymbol{v}\cdot\boldsymbol{a})\left(1+\gamma^2\left(\frac{v}{c}\right)^2\right),$$
$$=\gamma^2(\boldsymbol{v}\cdot\boldsymbol{a})\left(1+\frac{1}{1-\left(\frac{v}{c}\right)^2}\left(\frac{v}{c}\right)^2\right)$$
$$=\gamma^2\frac{\boldsymbol{v}\cdot\boldsymbol{a}}{1-\left(\frac{v}{c}\right)^2}\left(1-\left(\frac{v}{c}\right)^2+\left(\frac{v}{c}\right)^2\right)=\gamma^4\boldsymbol{v}\cdot\boldsymbol{a}.$$

8. Finally, substitute step 7 into step 6 and solve for $\boldsymbol{a}$,

$$\boldsymbol{a}_R=\gamma^2\left(\boldsymbol{a}+\frac{\gamma^2}{c^2}(\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{v}\right)=\gamma^2\left(\boldsymbol{a}+\frac{\gamma^2}{c^2}\frac{1}{\gamma^4}(\boldsymbol{v}\cdot\boldsymbol{a}_R)\boldsymbol{v}\right)$$
$$=\gamma^2\boldsymbol{a}+\frac{1}{c^2}(\boldsymbol{v}\cdot\boldsymbol{a}_R)\boldsymbol{v},$$
$$\boldsymbol{a}=\frac{1}{\gamma^2}\left(\boldsymbol{a}_R-\frac{1}{c^2}(\boldsymbol{v}\cdot\boldsymbol{a}_R)\boldsymbol{v}\right).$$

Steps 3–8 determined the relationships between the nonrelativistic and relativistic velocity and acceleration vectors,

$$\boldsymbol{v}_R=\gamma\boldsymbol{v}, \tag{B9a}$$

$$\boldsymbol{v}=\frac{1}{\gamma}\boldsymbol{v}_R, \tag{B9b}$$

$$\boldsymbol{a}_R=\gamma^2\left(\boldsymbol{a}+\frac{\gamma^2}{c^2}(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{v}\right), \tag{B9c}$$

$$\boldsymbol{a}=\frac{1}{\gamma^2}\left(\boldsymbol{a}_R-\frac{1}{c^2}(\boldsymbol{v}\cdot\boldsymbol{a}_R)\boldsymbol{v}\right). \tag{B9d}$$

Next, express the radial and circumferential components of the relativistic velocity vector and of the relativistic acceleration vector in terms of r and ϕ and their derivatives with respect to proper time. Start with expressing the rectangular components of the position vector of a point in terms of their associated radial and circumferential components as

$$\boldsymbol{r}=(r\cos\phi)\boldsymbol{i}_1+(r\sin\phi)\boldsymbol{i}_2. \tag{B10}$$

From Eqs. (B3a) and (B10), the radial and circumferential unit vectors are

$$\boldsymbol{n}_r=\frac{\boldsymbol{r}}{r}=\frac{1}{r}(r\cos\phi\boldsymbol{i}_1+r\sin\phi\boldsymbol{i}_2)=\cos\phi\boldsymbol{i}_1+\sin\phi\boldsymbol{i}_2,$$
$$\boldsymbol{n}_\phi=-\sin\phi\boldsymbol{i}_1+\cos\phi\boldsymbol{i}_2.$$

9. By differentiation with respect to proper time,

$$\frac{d\boldsymbol{n}_r}{d\tau}=(-\sin\phi\boldsymbol{i}_1+\cos\phi\boldsymbol{i}_2)\frac{d\phi}{d\tau}=\frac{d\phi}{d\tau}\boldsymbol{n}_\phi, \tag{B11a}$$

$$\frac{d\boldsymbol{n}_\phi}{d\tau}=-(\cos\phi\boldsymbol{i}_1+\sin\phi\boldsymbol{i}_2)\frac{d\phi}{d\tau}=-\frac{d\phi}{d\tau}\boldsymbol{n}_r. \tag{B11b}$$

10. From Eqs. (B4a) and (B11a),

$$\boldsymbol{v}_R=v_{Rr}\boldsymbol{n}_r+v_{R\phi}\boldsymbol{n}_\phi=\frac{d}{d\tau}(r\boldsymbol{n}_r)=\frac{dr}{d\tau}\boldsymbol{n}_r+r\frac{d\phi}{d\tau}\boldsymbol{n}_\phi,$$
$$v_{Rr}=\frac{dr}{d\tau},\quad v_{R\phi}=r\frac{d\phi}{d\tau}.$$

11. Finally, from Eqs. (B4b) and (B11) and the product rule,

$$\boldsymbol{a}_R=a_{Rr}\boldsymbol{n}_r+a_{R\phi}\boldsymbol{n}_\phi=\frac{d\boldsymbol{v}_R}{d\tau}=\frac{d}{d\tau}(v_{Rr}\boldsymbol{n}_r+v_{R\phi}\boldsymbol{n}_\phi),$$
$$=\left(\frac{dv_{Rr}}{d\tau}-v_{R\phi}\frac{d\phi}{d\tau}\right)\boldsymbol{n}_r+\left(v_{Rr}\frac{d\phi}{d\tau}+\frac{dv_{R\phi}}{d\tau}\right)\boldsymbol{n}_\phi,$$
$$=\left(\frac{d^2r}{d\tau^2}-r\left(\frac{d\phi}{d\tau}\right)^2\right)\boldsymbol{n}_r+\left(\frac{dr}{d\tau}\frac{d\phi}{d\tau}+\frac{dr}{d\tau}\frac{d\phi}{d\tau}+r\frac{d^2\phi}{d\tau^2}\right)\boldsymbol{n}_\phi,$$
$$=\left(\frac{d^2r}{d\tau^2}-r\left(\frac{d\phi}{d\tau}\right)^2\right)\boldsymbol{n}_r+\left(r\frac{d^2\phi}{d\tau^2}+2\frac{dr}{d\tau}\frac{d\phi}{d\tau}\right)\boldsymbol{n}_\phi,$$
$$a_{Rr}=\frac{d^2r}{d\tau^2}-r\left(\frac{d\phi}{d\tau}\right)^2,\ a_{R\phi}=r\frac{d^2\phi}{d\tau^2}+2\frac{dr}{d\tau}\frac{d\phi}{d\tau}.$$

12. Let $()'\triangleq d()/d\tau$. From the above, the relativistic velocity vector, the relativistic acceleration vector, and their components are

$$\boldsymbol{v}_R=r'\boldsymbol{n}_r+r\phi'\boldsymbol{n}_\phi, \tag{B12a}$$

$$v_{Rr}=r', \tag{B12b}$$

$$v_{R\phi}=r\phi', \tag{B12c}$$

$$\boldsymbol{a}_R=\left(r''-r\phi'^2\right)\boldsymbol{n}_r+\left(r\phi''+2r'\phi'\right)\boldsymbol{n}_\phi, \tag{B12d}$$

$$a_{Rr}=r''-r\phi'^2, \tag{B12e}$$

$$a_{R\phi}=r\phi''+2r'\phi'. \tag{B12f}$$

[1] M. P. Hobson, G. Efstathiou, and A. N. Lasenby, *General Relativity* (Cambridge University Press, New York, 2006).
[2] J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Cambridge University Press, New York, 2021).
[3] G. Gamow, *Gravity* (Courier Corporation, North Chelmsford, Massachusetts, USA, 2002).
[4] P. G. Bergmann, *The Riddle of Gravitation* (Courier Corporation, North Chelmsford, Massachusetts, USA, 1992).
[5] W. Rindler, *Essential Relativity: Special, General, and Cosmological* (Springer Science & Business Media, Berlin and Heidelberg, Germany, 2012).
[6] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Macmillan, New York, 1973).
[7] L. M. Silverberg, and J. W. Eischen, J. Spacecr. Rockets **60**, 1854 (2023).
[8] K. Schwarzschild, Sitzungsber. Preuss. Akad. Wiss. Berlin **7**, 189 (1916).
[9] L. M. Silverberg, and J. W. Eischen, Phys. Essays **33**, 489 (2020).
[10] L. M. Silverberg, and J. W. Eischen, Phys. Essays **34**, 548 (2021).
[11] I. Newton, *Newton's Principia: The Mathematical Principles of Natural Philosophy* (Daniel Adee, New York, 1846).
[12] J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Clarendon Press, Oxford, 1873).
[13] A. Einstein, Ann. Phys. **322**, 891 (1905).
[14] A. Einstein, Ann. Phys. **49**, 769 (1916).
[15] A. Einstein, L. Infeld, and B. Hoffmann, Ann. Math. **39**, 65 (1938).